Daily Numerical Skills

ASTR 170

"An innumerate citizen today is as vulnerable as the illiterate peasant of Gutenberg's time."
Lynn Steen

Employers want to hire people who understand numbers and can communicate effectively using numbers. The following skills will enable you to succeed in getting the jobs you want and will also help you be more productive and impressive.

From this following list a new skill will be assigned for each day of class. (Similarly for the Communication Skills) You are expected to use them immediately and to continue using each one throughout the remainder of the semester. Your weekly homework grade will be determined in part by your implementing these skills correctly. You are STRONGLY ENCOURAGED to seek help from your course instructors when any skill or concept is hard to understand or master!

#1. A "ratio" is a comparison of two numbers. In other words it is a "fraction" or a division of two numbers. The words "ratio," "fraction," and "percent" are all related.

Example: The ratio of the number one to the number two can be written in many different forms as listed below. Each of the following items is just a different way of expressing the same idea. You should become fluent with each technique.

$$1:2 = 1/2 = ½ = 1 \div 2 = 0.5 = 5/10 \text{ or } 50/100 = 50\% = 5 \times 10^{-1}$$

The word "percent" means 'what fraction out of 100?' so one-half means 0.5 or 50/100 or 50%.

Try it: Express the ratio of the numbers 3 and 9 using each of the above methods of expression.

#2. Become expert with the techniques of ratios, fractions, percentages, and proportions. They are all the same type of language and worth practicing.

Example: Last night I slept for eight hours. The ratio of my sleeping time to a full day is 8:24 or 8 hrs/24 hrs = 8/24 = 1/3 = 0.33 = 33% = 3.3×10^{-1} (notice the units cancel)

Note #1: In a ratio you should compare numbers using the same units. As a result, the units will cancel, and value of the ratio will be dimensionless. For example, 8 hours/24 hours does not equal 1/3 hours.

Note #2: You can also write eight hours as the following fraction of a day:
(8 hours / 24 hours/day) = 8/24 day = 1/3rd day = 0.33 day = 33/100 day = 33% of a day

Note #3: It is not correct to write that 4% of 1 = 0.4%. Instead, it would be correct to say
4% of 1 = 0.04 x 1 = 0.04 = 4 percent

#3. How many times larger is $10 than $1? The word "times" implies multiplication and a ratio.

Example: How many times more valuable is a ten-dollar bill than a one-dollar bill? The ratio is $10/$1 = 10, or ten times more valuable. It is true that the "difference" between a ten-dollar bill and a one-dollar bill is nine dollars, but that answer involves subtraction and is not what the question asks.

#4. Develop the habit of thinking "proportionally" in your head.

If a 5-ounce can of peanuts costs $0.45, then you might expect a can that weighs twice as much (i.e., a 10ounce can) to cost $0.90 - twice as much. In this example the cost is "directly proportional" to weight. In other words, if one characteristic increases, so does the other.

The term "inversely proportional" means the opposite. For example consider a car trip. If your speed increases, the travel time decreases.

Example: If a10-ounce can of peanuts costs 90 cents and a 4-ounce can costs 45 cents, which item is more cost-effective? Since 90 cents is twice 45 cents, you would expect that the first can would contain two times more peanuts than the second can. Instead, the first one contains more than eight ounces (2.5x more) so it is a better deal.

#5. Be sure you understand what exponents <u>mean</u> - then you can handle powers of ten properly. In this course, you must memorize the meaning of thousand, million, billion, and trillion. An exponent simply tells us how many times a number is multiplied by itself.

Numbers expressed this way can be a helpful shorthand. For example, the notation 10^3 (also written as 10^3) is 10x10x10 = 1000. Similarly, 2^3 equals 2x2x2 = 8.

Negative exponents indicate an “inverse” or upside-down situation. For example, the number 10^{-3} (or, 10^3) means $1/10^3$ or 1/(10x10x10) = 1/1000 = 0.001.

<u>Incorrect</u>: 9,500,000 does not equal 9.5^6. The statement should be written as 9.5×10^6 (or, 9.5 x 10^6) which means 9.5 million. The notation of 9.5^6 means 9.5x9.5x9.5x9.5x9.5x9.5 which equals 735091.8906 – a big difference!

Memorize the following language, powers of ten, and prefixes:

thousand = 10^3 or one kilo-dollars
million = 10^6 or one mega-dollars
billion = 10^9 or one giga-dollars
trillion = 10^{12} or one tera-dollars

Down the list, each of these numbers is one-thousand times larger than the previous one.

<u>Incorrect</u>: “The energy released by the impact that created Meteor Crater was about 10^4 megatons of TNT, which equals 10^6 kilotons. Therefore, the impact was the equivalent of 6 atomic bombs.”
<u>Answer</u>: The number 10^4 means 10x10x10x10 = 10,000. “Mega” means million (i.e., 10^6). Therefore 10^4 megatons equals 10^{10} tons. Since ‘kilo’ means 1000, 10^4 megatons also equals 10^7 ktons. The Hiroshima atomic bomb releeased an energy of 10 ktons (i.e., 10,000 tons), so Meteor Crater’s energy is equivalent to 10^7 ktons/10^1 ktons= 10^6 atomic bombs.

#6. Become proficient in “powers of ten.” This notation is the basis of our monetary system and is necessary for your personal accounting and to be an informed citizen.
<u>Example</u>: Our nation’s annual debt is more than one trillion (10^{12}) dollars. To reduce the debt, would it help much to save one billion (10^9) dollars? No, because one billion is one thousand times (10^3) less than one trillion. Compared to one trillion, one billion is $10^9/10^{12}$ = 1/1000 = 0.1%, in other words, a very small fraction. You have been told to memorize the meaning of thousand, million, billion, and trillion and their associated powers of ten.
<u>Example</u>: “The earthquake caused estimated property damage of $400 million, or more than $8 billion in today’s dollars.” How many times less valuable is today’s dollar?
[Deck of playing cards “*1906 Earthquake: Facts and Stories*”; 2005; Golden Gate National Parks Conservancy]

<u>Incorrect</u>: 10^11 – 10^3 = 10^8 – In words this statement means one-hundred billion minus onethousand equals one-hundred million. That answer is not even close to being correct because the number 1000 is much, much smaller than 100 billion (one million times smaller).
<u>Correct</u>: 10^11 – 10^3 is still about 10^11

#7. Your numbers do not always need to be “precise” with lots of decimal points. Stick to round numbers unless otherwise asked.
<u>Example</u>: Precise numbers with lots of digits and decimal places are not always needed. For example, from homework, the precise number of Earths that would fit between the Earth and the Sun is 11,538.46. If you realize that about 10,000 Earths would fit, the main purpose is served and also meets the stated requirement. Is the extra 0.46 of an Earth really significant in this context; do the Earth and Sun really have an exact size given their fuzzy atmospheres, etc.? Similarly, do we need to know the US national debt to

the penny out of trillions of dollars: $13,446,515,029,743.15? On the other hand, scientists and engineers need to know the speed of light very precisely: 299,792,458 meters/sec.

#8. Keep track of the "units" of measurement!

Example: From homework, "The distance between our galaxy and the next nearest large galaxy is about two million light-years. The radius of the Milky Way is about 50,000 light-years. How many Milky Ways would fit side-by-side between these two galaxies?" Can you find at least three things wrong with the following answer?

2,000,000/50,000 = 40 light-years = 4.0^1

Corrected: 2,000,000 light-years/100,000 light-years = 20 =
= 2.0 x 10^1

#9. What does the question ask? "Look before you leap!" Be sure you understand the terminology, even if you have to double-check a vocabulary word you may think you know. It is usually helpful to make a drawing!

Example: A question in homework asked you to calculate the number of Earths that could fit side-byside across the Sun. Would that mean the Sun's "radius," "diameter," "circumference," or "volume"?

#10. Embed numbers in your writing to strengthen your explanations and to make your arguments more effectively. Numbers are your friends!

Example: Which statement is more informative? (1) "My invention can save the average homeowner $100 a month on his/her utility bill." (2) "My invention can save the average homeowner a lot of money."

#11. What is an angle? What are the units of measurement?

Example: An angle is the inclination between two lines which meet each other at a common point. As the angle between them increases, one line "arcs" around that common point. A full rotation between these lines yields a rotation, or arc, of 360 degrees and produces a circle. Astronomers subdivide one degree into 60 smaller parts called arcminutes and also each arcminute into 60 arcseconds.

#12. The square root of a number is that special value that, when multiplied by itself, gives the number.

Example: $\sqrt{9} = 9^{1/2} = 3$

If we moved Earth to a distance (a) from the Sun of 100 AU, what would be its new orbital period (P)? We can rewrite Kepler's Third Law ($P = a^{3/2}$) as follows $P = a^{3/2} = (a^3)^{1/2} = (a^{1/2})^3$. So, for a = 100 AU, $P = (10)^3$ = 1000 years.

#13. What do we mean by "inverse-square"?

Example: Newton's law of gravity is written as $\mathbf{F = GMm/d^2}$. You can make use of 'proportional thinking' to understand the story this equation tells. The force of gravity depends directly on the masses such that doubling one of the masses doubles the force. However, the force depends inversely (i.e., upside down) on the square of distance between two objects. If we move the objects 3x farther away (to 3d), the quantity $1/d^2$ becomes $1/(3d)^2 = 1/9d$ or 1/9th of its earlier value, so the force decreases 9x.

#14. What does your answer mean? Is it reasonable, surprising, absurd?

Example: Concerning a previous homework problem, the Earth's radius is 6.4×10^6 meters, is it reasonable to think that its diameter is 12.8×10^{12} meters? Restate those two numbers in words (i.e., 10^6 vs. 10^{12}) and determine if one is twice the other.

#15. Be able to do "scaling" in your head.

Example: Area is two-dimensional. For example, the area covered inside a circle is $A = \pi r^2$. What happens if the radius triples, going from "r" to "3r"?

Solution: Area $= \pi(3r)^2 = \pi\,(9r^2) = 9 \times \pi r^2 = 9\pi r^2$

The new value is nine times the original area of πr^2, where r = 1.

Example: The force of gravity depends on the inverse-square of the separation between two masses: $F = GMm/d^2$. What happens if the separation increases five times, going from "d" to "5d"?

Solution: $F = GMm/(5d)^2 = 1/25 \times GMm/d^2$.

So the new force is 25x less than the original.

#16. A sphere has "surface area."

Example: The "area" of any object is two-dimensional and is measured in "square-units." The surface area (A) of a sphere is $A = 4\pi(\text{radius})^2$. Imagine two bubbles, one with radius = r and a larger one with a radius = 3r. The larger bubble has a surface area of $A = 4\pi(\text{radius})^2 = 4\pi(3r)^2 = 4\pi(9r^2)$. The smaller bubble would have an area of $4\pi(r^2)$. So, if the radius of a bubble triples, its surface area increases 9x.

#17. Percentage: "More or less"

Be careful with words and phrases using "percent more" or "percent less." Example: Imagine you have a savings account. Long ago you deposited $100. After years of gaining interest, it is worth $400 today. Incorrect: It is not correct to say that the account is worth "1/4th more" today. Similarly, it is not worth "25% more" today.

Correct: It is OK to state that the original account was worth 1/4th = 25% of its value today. It is also correct to say that today's value is four times larger.

#18. "A 200% increase."

Correct: If some quantity increases by 100%, then its new value is 200% of the original amount (i.e., 100% of initial + 100% increase = 200% of initial.) In other words the amount has doubled.

Example: if your weight was 50 pounds and it increased by 100%, then your present weight is 100 pounds. If your original weight increased 300%, then you would weigh 200 pounds: The original 50 + another 50+50+50 = 200 pounds.

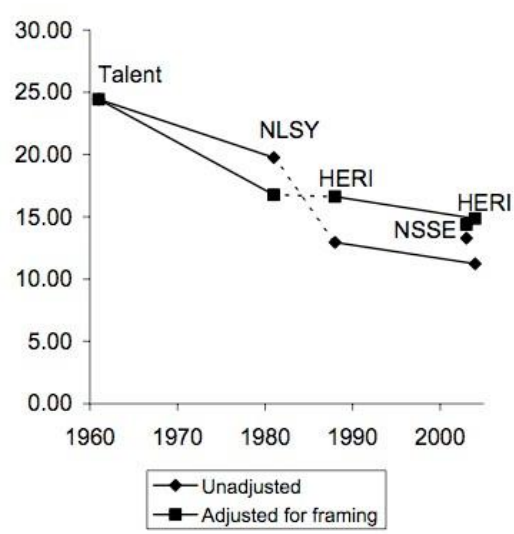
Example: In the example above for Skill #16, the value of your savings account today is 300% larger than the original amount, or 4x the original amount.

Question: If 300 is 50% bigger than 200, is the number 200 50% less than 300?

#19. Let graphs tell you a story. First, look for the important "characters."

Example: The following graph shows the trend of "study hours" for full-time college students with year from 1960 to 2004. It shows that students have been spending steadily fewer hours studying since 1960 and that today's students spend almost two times less studying than students ~50 years ago.

Figure 1
Average Study Hours



#20. Efficiency

Example: If a telescope's main mirror reflects 90% of the light that hits it, then what percentage is left after two reflections?

Answer: The first mirror reflects 90% of the original light. The second mirror reflects 90% of that amount, so the remaining light is 0.9 x 0.9 = 0.81 = 81% of the original.

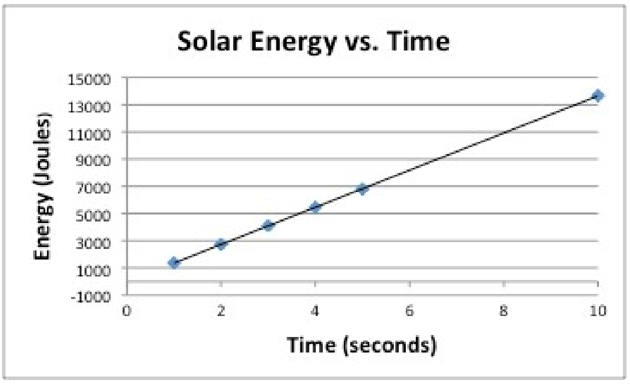
#21. Converting from 'mega' to 'kilo' (See Skill #8 for review.)

Example**:** The energy released in the impact that created the crater in Germany was equivalent to ~100 megatons TNT. How many kilotons of TNT? How many A-bombs at 10 ktons each?

Answer: 'Kilo" means thousand, and 'mega' means million. Therefore, there are 1,000 kilotons in one megaton. The German crater's energy was equivalent to 100 x 1,000 kilotons or 100,000 (i.e., 10^5) ktons and 10,000 (i.e., 10^4 A-bombs).

#22. Linear growth

Example: Exposure to sunlight increases with time. The Earth's surface receives solar energy at the rate of 1.3 kWatts/m^2 or 1366 joules per sec for every square-meter. How much energy after 2 sec, 5 sec, 10 sec, etc.?



#23. Exponential growth: "Involving an exponent."

Example: How many times can you fold a piece of paper in half? How many times thicker would it be after 7 folds?

The number increases exponentially as the number two to the exponent of the number (n) of folds, or 2^n. The sequence (n) of folds produces the following thicknesses of paper:

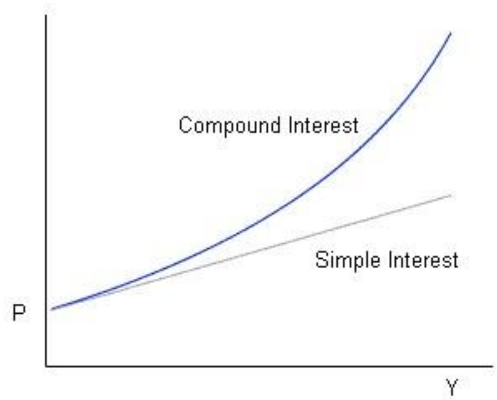
(n=1) 2; (n=2) 4; (n=3) 8; (n=4) 16; (n=5) 32, (n=6) 64; (n=7) 128; (n=8) 256; (n=9) 512; … 2^n

In class we also discussed the "half-life" of a radioactive element as an example of exponential decay.

#24. Exponential growth

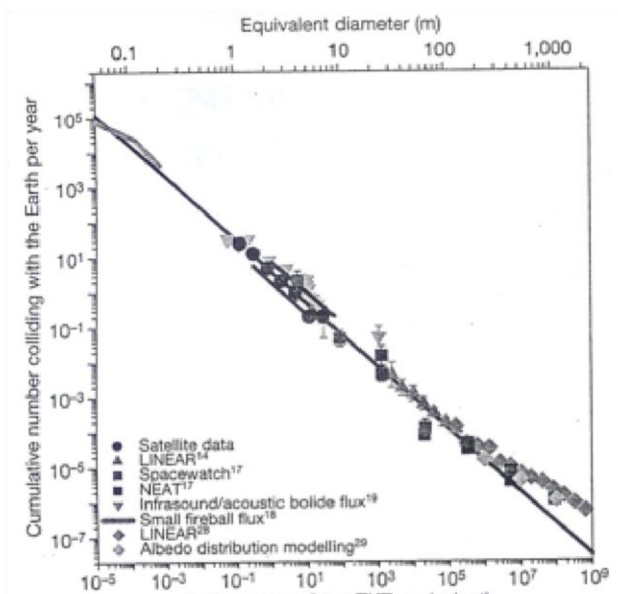
Example: If a quantity changes by a fixed fraction per unit of time (e.g., 10% per year), then the quantity grows exponentially. For instance, imagine you start now to save money for your retirement, and you save $1,000 per year at 5% interest. Every year the interest is accumulated and itself earns interest the next year (i.e., compounded). Forty years later your money will total $135,000, much more than the $40,000 you deposited!

In contrast, if the interest is simply added to the original without being compounded, the value would steadily (linearly) increase. In Skill #21, we discussed "linear growth."



#25. Data and estimates of error. Interpolation.

Example: The diagram below shows the results of real measurements from several different sources. Homework #18 asked you to determine how often the Earth is impacted by an object releasing an energy of 10 Ktons TNT (i.e., the Hiroshima atomic bomb)



Starting at 10 Ktons (i.e., 10^1) on the horizontal axis, extend a vertical line to the **midpoint** of the data shown and then determine the corresponding vertical coordinate value, a frequency of 10^0, meaning such an event happens about once per year. The data not always agree because all measurements have errors and biases, etc. So, the idea is to locate the midpoint of the data, as though you were taking an average. Don't use the uppermost or lowermost values.

In the range of diameters from 0.2-1 meters, no data is shown but a solid line indicates what we expect to occur based on results from >1 meter diameters and <0.1 meters. We say that we have "**interpolated**" in between the data. Of course, it's always best to conduct actual measurements but the line shows what we expect.

#26. Handling exponents.

Example: The radius of Earth is

6378 km $= 6.4 \times 10^3$ km $= 6.4 \times 10^6$ m $= 6.4$ million meters

Which of the following is (are) Earth's diameter?

a. 12.8×10^{12} m (INCORRECT – don't double the exponent! That is like multiplying by one million!) b. 12.8×10^6 m (CORRECT)

c. 12.8 million meters (CORRECT)

#27. Pay Attention to "Error Estimates."

As mentioned in Skill #24, every measurement has inherent errors associated with the measurement process, often imposed by Nature itself. In the following News Report, it is correct to state that President Clinton "slipped against Bob Dole"? Why or why not?

1996 News Report

President Clinton, hit by bad publicity recently over FBI files and a derogatory book, has slipped against Bob Dole in a new poll released Monday but still maintains a 15 percentage point lead.

The CNN/USA Today/Gallup poll taken June 27-30 of 818 registered voters showed Clinton would beat his Republican challenger if the election were held now, 54 to 39 percent, with seven percent undecided. The poll had a margin of error of plus or minus four percentage points.

A similar poll June 18-19 had Clinton 57 to 38 percent over Dole.

The recent poll had a so-called "margin of error" of 4% (plus of minus). This characteristic means that the numbers are never exact and could be wrong by a certain amount. Specifically, it means that if you obtained 100 similar polls, 95% of the time the results are likely to vary by plus or minus 4%.

For example, in the poll of June 18-19, Clinton had 57 percent of the vote. Allowing for expected error, Clinton 's number could have ranged between 53-61, and Dole's could have ranged between 34-42 percent.

The change from June 18-19 to June 27-30 is not significant compared to these errors, so it is not correct to state that Clinton had "slipped" in a new poll.

#28. Square-root: Remember that words can also be numbers! (A followup to Skill #11)

Example**:** The star "α Centauri A" appears 81 billion times fainter than the Sun. How many times farther away is α Centauri A?

When you take the square-root of 81 billion, remember that the word "billion" refers to the number 1,000,000,000. So you need to take the square-root of 81,000,000,000. The answer is not 9 billion but instead is about 300,000. That's a big difference!

#29. Be careful entering powers of ten on your calculator.

What does this frequency mean: 1.5 E+14 Hertz? It means 1.5×10^{14} Hertz.

On your calculator type:

a. 1.5 EXP 14 or

b. 1.5 EE 14

Do not type: (1.5 x 10 EE 14) because that procedure yields 1.5×10^{15} Hertz

#30. Do the following statements demonstrate correct use of exponents?

Examples: $4.3 \times 10^{-6} = 4.3000000$

$2.7 \times 10^6 = 2.7000000$

2.7 E11 = 2.7

Improved: Exponents do not indicate the precision of a number, i.e., not the number of figures following the decimal point. Instead they indicate the overall size ("magnitude") of the entire number.

$4.3 \times 10^{-6} = 4.3 \ / \ 1{,}000{,}000. = 0.0000043$

$2.7 \times 10^6 = 2.7 \times 1{,}000{,}000 = 2{,}700{,}000.$

2.7 E11 = 2.7 x 100,000,000,000. = 270,000,000,000.

#31 A sphere has "volume."

Example: Volume is three-dimensional. The volume (V) of a sphere is $V = 4\pi r^3/3$. Pretend one bubble has three times the radius of another's. How many times larger is the volume? If 'r' triples, then 'r^3' becomes 3x3x3 = 27 times larger.

#32. Percentage growth

Example: Your bank account: (2005: $1,000) and (2010: $1,350)
Growth = $350
Percentage growth: $350/$1000 = 0.35 = 35%
Percentage growth per year = 35%/5 years = 7% per year

#33. "Scaling" (A followup to Skill #15)

Example**:** Among the 1000 stars in our "solar neighborhood," we expected 0.4 Earth-mass planets to orbit solar-type stars in the habitable zone. How many such planets would we expect within our entire Milky Way galaxy of ~100 billion stars?

Answer**:** The number 100 billion (10^{11}) divided by one-thousand (10^3) equals 100 million (10^8) times more stars. So, the number of expected Earth-mass planets would equal
[100 billion] / [1000] times 0.4 = 4 x 10^7 = 40,000,000 = 40 million

"Scaling" means either to multiply or to divide by some number. You are performing "scaling" when you select "enlarge" on a copy machine or when you "zoom" into, or out of, a Google map. We have probably all built "scale models" where a very large object (e.g., an airplane, car, house, etc.) is reduced in size to fit on a table top or in our hands.

#34. More scaling: "Parts per million"

Example: The amount of carbon dioxide in our atmosphere is ~385 ppm and is increasing at ~15 parts per million (ppm) per decade. That statistic means that for every one million molecules of air, about 385 are carbon dioxide. Similarly, the Environmental Protection Agency has set the water quality standard of 0.340 ppm of arsenic in our drinking water.

#35. Averaging I. When using an "average," be careful to understand the relevance of the result.
(http://conceptualmath.org/philo/staterr.html)

Example: Which group is better off?

Group A

- dead
- in jail
- earns $30,000
- earns $50,000

Group B

- earns $10,000
- earns $20,000
- earns $30,000
- earns $50,000

Solution**:** Group "B" has an average income $27,500, about 2/3 the average income of Group "A."

#36. Averaging II. When using an "average," watch for "outliers" that may not belong in the sample.

Example: Which group has the higher average salary?

Group A

- earns $10,000
- earns $20,000
- earns $10,000
- earns $30,000
- earns $1,000,000
- earns $40,000

Group B

- earns $10,000
- earns $30,000

Questions: How could you make these comparisons more fair and relevant?

Comment: Do averages really represent the individuals? Imagine what the average income data will look like for any group that includes Bill Gates. During the first decade of the 21st century, the average income rose, but the median income stagnated. What did this result mean?

#37. Averaging III. How relevant is an "average" to a specific person?
(http://conceptualmath.org/philo/staterr.html)

Example: How relevant is “life expectancy” to a specific individual?
Imagine a small town with a high infant mortality rate. The recorded deaths have occurred at these ages: Ten infants died in their first year, and four adults at the ages 60, 70, 75, and 80. This town’s life expectancy will be calculated as 20, by using an average. Should the young start worrying as they approach the age of 20?

Question: How could you make this calculation more fair and relevant?

Comment: In many locations and times in history life expectancy has been reported to be less than 30. How should we interpret this? Did most people really die when they were 30? How would this have affected families? How old would most children have been when their parents died? Who would have raised the children?

#38. Averaging IV. When using an “average,” be careful of small numbers in your groups.

Example: Which generation earned the higher salary?

Generation #1
- A parent of 1 earning $60,000
- A parent of 5 earning $20,000

Generation #2
- One grown child earning $60,000
- Five grown children earning $20,000

Solution: The average income dropped from $40,000 to $27,000 from one generation to the next, yet the offspring grew up to earn the same as their parents. Is correct to say that incomes dropped? Is it correct to say that incomes stayed the same?

Question: How could this data be presented in a clearer way than average income? Consider using the “**median**” salary which would show the value that separates the higher half of the data from the lower half.

#39. Reprise to #24,27: Pay attention to error estimates.

Example: The measured age of the Universe is (13.7 ± 1%) billion years. How well do scientists really know the age of the Universe? Does this value mean 13.7 ± 0.01 or 13.7 ± 0.137?

Answer: Every scientific measurement should be accompanied with a rigorous statement of the associated error. Typically, scientists quote errors as plus or minus three “standard deviations” (i.e., $\pm 3\sigma$, also known as “3 sigma”). This method indicates that we are 99.7% confident that the actual value lies within this total range.

One percent of 13.7 billion years equals 0.137 billion years. If our measurements are correct, then the age of the Universe must be within ±3(0.137) = ±0.411 or between 13.289 and 14.111 billion years.

#40. Numerical Shortcuts: Do it in Your Head !

Question: What is the exposure time in hours of 342 exposures each lasting 30 minutes?

Answer: You could do this problem in your head is you realize that 30 minutes is one=half of an hour. So 342 x ½ = 171 hours. Yes, you could also do the same problem with a calculator

(342 exposures) x (30minutes/exposure) x (1 hour/60 minutes)

There is power in doing calculations, even approximate ones, in your head because you are likely to impress your employer, solve your problem quicker, and you can do the problem without reaching for any kind of technology.

#41. Percentage Language

Question #1: If 300 is 50% bigger than 200, is 200 50% less than 300?

Answer: No. Fifty percent of 300 equals 150, so 50% less than 300 equals 150.

Question #2: In 1990, pretend that Product-A had 10% of market share and now it has 15%. Is that a 50% increase or a 5% increase?

Answer: An increase in what quantity? We should be more careful with language in such statements because the reader may misinterpret the meaning. For example, it is possible that the market declined since 1990, so there may be less of Product-A sold today. The best technique is to re-state the problem more specifically.

#42. Percentage Language (continued)

Question: Is the following statement from a student true or false: *"The size of the images went from 8% the size of the Full Moon to 10% the size of the Full Moon, a 125% increase in size."*?

Answer: The statement is confusing. A 125% increase means an increase of 1.25 times 8% or a 10% **increase** relative to the initial image, producing a new result of 8+10% = 18% of the Full Moon. It is true that the image got 125%, or 1.25 times, larger.